

Baroclinic equivalent and nonequivalent barotropic modes for rotating stratified flows

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(Dated: February 15, 2009)

In strictly speaking, all the natural phenomena on the earth should be treated under rotating coordinate. The existence of baroclinic nonequivalent barotropic laminar solution for rotating fluids is still open though the laminar solutions for the irrotational fluid had been well studied. In this letter, all the possible equivalent barotropic (EB) laminar solution are firstly explored and all the possible baroclinic non-EB elliptic circulations and hyperbolic laminar modes are discovered. The baroclinic EB circulations (including the vortex streets and hurricane like vortices) possess rich structures because either the arbitrary solutions of arbitrary nonlinear Poisson equations can be used or an arbitrary two-dimensional stream function is revealed. The discovery of the baroclinic non-EB modes disproves a known conjecture. The results may be broadly applied in atmospheric and oceanic dynamics, plasma physics, astrophysics and so on.

PACS numbers: 47.32.-y, 47.32.ck, 47.55.Hd

1. Introduction. It is known that both the planetary rotations and stable vertical density stratification are important for the fluid motions in atmospheres and oceans. The effect of rotation and stratification are the most important features that distinguish fluid flow in the atmosphere and ocean. The flows in the rotating stratified fluids exhibit rich phenomena especially on the circulation vortices [1, 2] like the Jupiter's Red Spot, tropospheric cyclones, hurricanes[3, 4], tornados, stratospheric polar vortices, oceanic Gulf Stream rings, atmospheric blockings which are mainly responsible for many kinds of meteorological disasters such as the floods, droughts and snowstorms etc. [5, 6]

For the usual irrotational fluid, the laminar solution is studied quite well. However, for the rotational fluid, there are many open problems on laminar solutions.

Recently, a steady baroclinic laminar model

$$\begin{aligned} uu_x + vu_y - fv &= -p_x, \quad uv_x + vv_y + fu = -p_y, \\ p_z &= -\rho, \quad u_x + v_y = 0, \quad u\rho_x + v\rho_y = 0, \end{aligned} \quad (1)$$

where f is the Coriolis parameter, p is the pressure perturbations divided by a mean density ρ_0 and ρ is the density perturbation scaled by ρ_0/g , u and v are horizontal velocities while the vertical velocity w has been dropped out because its weakness, is developed as the late-time equilibrium state in the free decay of rotating stratified.

To derive the model (1), the author has hypothesized that the formation mechanism for coherent structures in rotating stratified flows is fundamentally baroclinic. However, to find exact baroclinic solutions in fluid dynamics is very difficult and there is little progress in this direction. In Ref. [7], one type of special barotropic tilting vortex solution and four special types of baroclinic equivalent-barotropic (EB) are obtained. Basing on the fact that all the known solutions are either barotropic or baroclinic EB, a conjecture is proposed.

Conjecture: *Baroclinic solutions to (1) are always EB.*

Now, important questions are: How to find possible baroclinic modes of (1)? Is the conjecture correct?

2. Baroclinic EB modes. Here, we try to find *all* the possible Baroclinic EB modes of the baroclinic laminar model (1).

From the incompressible condition, $u_x + v_y = 0$, we can introduce stream function ψ as

$$u = -\psi_y, \quad v = \psi_x. \quad (2)$$

After introducing the stream function as in (2), five equations shown in (1) are reduced to two equations for the single function ψ ($K \equiv \frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_y^2$, $\zeta \equiv \psi_{xx} + \psi_{yy}$, $J(a, b) \equiv a_x b_y - a_y b_x$)

$$J(\psi, K_z) - (\zeta + f)J(\psi, \psi_z) = 0, \quad (3)$$

$$J(\psi, \zeta) = 0. \quad (4)$$

Eq. (3) is just the last equation of (1) while Eq. (4) is the consistent condition of the first two equations of (1), i.e., $p_{xy} - p_{yx} = 0$. Whence the stream function ψ is solved out from (3) and (4), the velocity components is obtained immediately from (2), the pressure can be solved out from the consistent equations, the first two equations of (1) while the density is only a simple differentiation of the pressure with respect to z .

Definitions: The fluid is barotropic if density is a function of pressure only, that is, isobaric surfaces and isopycnal surfaces coincide; otherwise, the fluid is baroclinic. A baroclinic flow is EB if the streamlines on each plane align vertically or, equivalently, if the horizontal velocity vector does not change direction vertically. More clearly, the fluid is barotropic iff the pressure of (1) is a function of $z + h$ with an arbitrary function $h \equiv h(x, y)$ while the fluid is called baroclinic EB of (1) iff

$$\psi_x = F\psi_y \quad (5)$$

for arbitrary $F \equiv F(x, y)$.

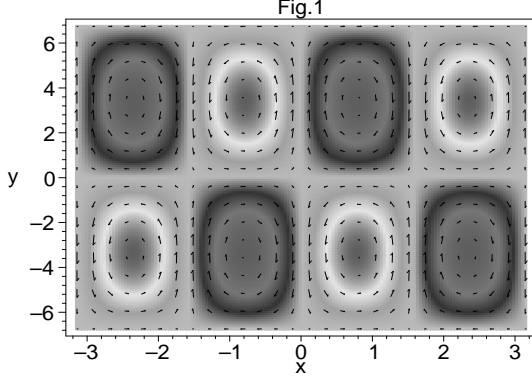


FIG. 1: The density plot of the vortex street solution (10) and the vector field plot of the corresponding velocity field expressed by (12) and (13) with the parameter selections $a = \frac{1}{8}$, $b = 2$, $c = \frac{1}{2}$.

From (5) and (3), it is easy to find that the only two possible cases of baroclinic EB, (A) $\psi_{yz} = 0$ and (B) $F_y + FF_x = 0$.

(A) *Baroclinic EB with an arbitrary nonlinear Poisson flow.* For the $\psi_{yz} = 0$ case, we have the stream function

$$\psi = \phi(x, y) + \psi_0(z), \quad (6)$$

with $\psi_0(z)$ being an arbitrary function of z while $\phi \equiv \phi(x, y)$ is a solution of an arbitrary nonlinear Poisson equation

$$\phi_{xx} + \phi_{yy} = g(\phi), \quad (7)$$

where $g(\phi)$ is an arbitrary function of ϕ . Whence the Poisson equation (7) is solved, the other quantities can easily be found. The results read

$$u = -\phi_y, \quad v = \phi_x, \quad (8)$$

$$p = \frac{1}{2}\phi_y^2 + \int \phi_x \phi_{yy} dx + f\phi + \phi_0(y) + p_0(z), \quad (9)$$

where $p_0(z)$ is an arbitrary function of z while $\phi_0(y)$ should be appropriately fixed such that the first two equations of (1) are compatible.

In Fig. 1, a special vortex street solution $((m, n) \equiv (a\frac{c}{b}, a\frac{b}{c}))$,

$$\psi = \phi = 4 \arctan(\operatorname{asn}(bx, m)\operatorname{sn}(cy, n)), \quad (10)$$

where a , b and c are constants while $\operatorname{sn}(bx, m)$ is the standard Jacobi elliptic function with modula m , is shown with the parameter selections $a = \frac{1}{8}$, $b = 2$, $c = \frac{1}{2}$.

Corresponding to the solution (10), the arbitrary function of Poisson equation is fixed as

$$g(\phi) = -(b^2 + c^2)(1 + a^2) \sin(\phi), \quad (11)$$

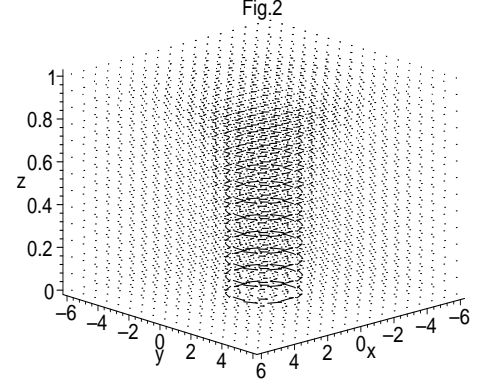


FIG. 2: A 3-dimensional vortex solution described by the vector velocity field (18)-(19).

and the other physical quantities are

$$u = -\frac{4a\operatorname{csn}(bx, m)\operatorname{cn}(cy, n)\operatorname{dn}(cy, n)}{1 + a^2\operatorname{sn}^2(bx, m)\operatorname{sn}^2(cy, n)}, \quad (12)$$

$$v = \frac{4ab\operatorname{cn}(bx, m)\operatorname{dn}(bx, m)\operatorname{sn}(cy, n)}{1 + a^2\operatorname{sn}^2(bx, m)\operatorname{sn}^2(cy, n)}, \quad (13)$$

$$p = f\phi - 8b^2 \frac{c^2\operatorname{sn}^2(bx, m) + b^2\operatorname{sn}^2(cy, n)}{1 + a^2\operatorname{sn}^2(bx, m)\operatorname{sn}^2(cy, n)} + g, \quad (14)$$

where $g \equiv g(z)$ is an arbitrary function of z .

(B) *Baroclinic EB symmetric circulations.* For the $F_y + FF_x = 0$ case, it is straightforward to prove that the only possible modes are

$$\psi = \psi_0, \quad r \equiv c_1(x^2 + y^2) + c_2x + c_3y, \quad (15)$$

$$u = -\psi_{0r}(2c_1y + c_3), \quad v = \psi_{0r}(2c_1x + c_2), \quad (16)$$

$$p = 2c_1 \int \psi_{0r}^2 dr + f\psi_0 + p_0(z), \quad (17)$$

where c_1 , c_2 and c_3 are arbitrary constants while $p_0 \equiv p_0(z)$ and $\psi_0 \equiv \psi_0(r, z)$ are arbitrary functions of the indicated variables.

It is clear that there exist abundant symmetric circulation modes ($c_1 \neq 0$ in (15)) and jet modes ($c_1 = 0$ in (15)) because the stream function is an arbitrary function of two variables r and z . The richness of the symmetric circulations for the rotational fluids is natural as one had observed in both the oceans and the atmosphere. Actually, both in the atmosphere and in the oceans, there are also many kinds of nonsymmetric circulations.

In Fig. 2, a special second type of baroclinic symmetric EB mode is plotted for the velocity field ($r \equiv x^2 + y^2 - 2$)

$$u = 2(z - 1)y\operatorname{sech}[(1 - z)r]\operatorname{sech}(1 - z), \quad (18)$$

$$v = 2(1 - z)x\operatorname{sech}[(1 - z)r]\operatorname{sech}(1 - z) \quad (19)$$

which is related to the stream function solution (15)

$$\psi = \operatorname{sech}(1 - z) \arctan\{\sinh[(1 - z)r]\}. \quad (20)$$

Correspondingly, the pressure has the form

$$p = 2(1 - z) \tanh[(1 - z)r]\operatorname{sech}^2[(z - 1)r] - 2f\operatorname{sech}(1 - z) \arctan\{\exp[(z - 1)r]\} + p_0, \quad (21)$$

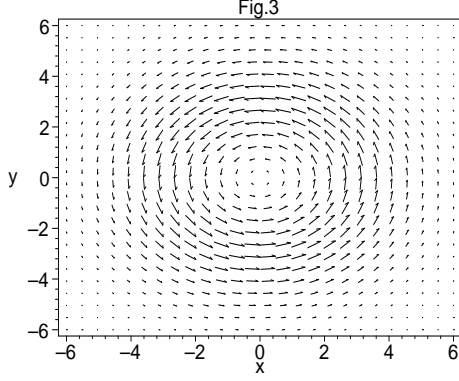


FIG. 3: The hurricane like structure which is the bird's eye view of Fig. 2.

where p_0 is still an arbitrary function of z . The overlooking form of Fig. 2 has the form of Fig. 3 which exhibits a hurricane like circulation form with a hurricane eye.

3. *Baroclinic non-EB elliptic and hyperbolic modes.* To find a nonsymmetric circulation, we restrict ourselves to find elliptic or hyperbolic modes of (1). For an elliptic or hyperbolic mode is defined as its stream lines are elliptic and/or hyperbolic curves. In other words, the stream function ψ has the form

$$\psi = \psi(a_1(z)(x - x_0(z))^2 + a_2(z)(y - y_0(z))^2, z), \quad (22)$$

with $a_1(z)a_2(z) > 0$ for elliptic and $a_1(z)a_2(z) < 0$ for hyperbolic modes.

Substitute (22) into (4), one can easily find

$$\psi_\xi \psi_{\xi\xi} [a_2(z) - a_1(z)][y - y_0(z)][x - x_0(z)] = 0, \quad (23)$$

where $\xi \equiv a_1(z)(x - x_0(z))^2 + a_2(z)(y - y_0(z))^2$.

From (23), we know that the only case is $\psi_{\xi\xi} = 0$ for nonsymmetric ($a_2(z) \neq a_1(z)$) modes, i.e.,

$$\psi = a_1(z)(x - x_0(z))^2 + a_2(z)(y - y_0(z))^2 + \psi_0(z) \quad (24)$$

which lead to that the equation (3) is correct only for the two nontrivial cases:

Case 1. Baroclinic elliptic or hyperbolic non-EB modes with rotational shape as the height z changes:

$$\psi_\pm = \pm \frac{1}{2h} g_\pm \eta_\pm^2 + \psi_0 \quad (25)$$

$$u_\pm = \frac{1}{h} g_\pm (y - y_0), \quad v = g_\pm h (x - x_0), \quad (26)$$

$$p_\pm = p_0 + \frac{1}{2} g_\pm \left(g_\pm \eta_\pm^2 \pm f \frac{\eta_\pm^2}{h} \right), \quad (27)$$

where h , ψ_0 and p_0 are arbitrary functions of z , $\{x_0, y_0, c_1\}$ are arbitrary constants while $\eta_\pm^2 \equiv (y - y_0)^2 \pm h^2(x - x_0)^2$, $\eta^2 \equiv (x - x_0)^2 + (y - y_0)^2$, $g_+ \equiv c_1 - \text{arctanh}(h)$, $g_- \equiv c_1 - \text{arctan}(h)$. The upper sign is related to the baroclinic elliptic circulation while the lower sign corresponds to the baroclinic hyperbolic wave case.

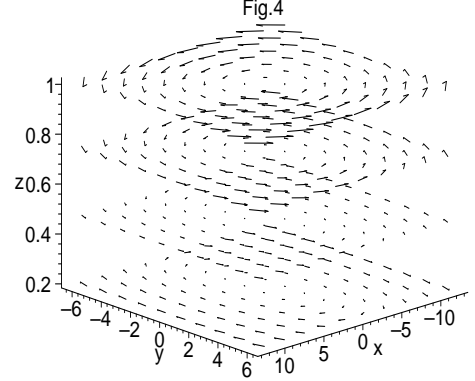


FIG. 4: Baroclinic elliptic circulation for the vector velocity field described by (28)-(29).

Fig. 4 displays a special structure for a baroclinic elliptic circulation with the velocity field

$$u = \frac{y}{10}(1 + z^2) \left(\text{arctanh} \frac{1}{1 + z^2} - \frac{3}{2} \right), \quad (28)$$

$$v = \frac{x}{10(1 + z^2)} \left(\frac{3}{2} - \text{arctanh} \frac{1}{1 + z^2} \right) \quad (29)$$

which corresponds to the selections $h = \frac{1}{1+z^2}$, $c_1 = \frac{3}{2}$, $x_0 = y_0 = 0$, $f = \frac{1}{10}$ in (26).

From (25)-(27), we find that the baroclinic elliptic circulation possesses some interesting properties. (i) The circulation center is independent of the height z . (ii) The length of the elliptic axes are changeable as z and then the circulation shape is rotated as z changes. (iii) All the quantities, the stream function, the velocity field and the pressure and density, possess elliptic distributions. (iv) The rotation direction of the vortex may be changeable if $g_+ = c_1 - \text{arctanh}(h) = 0$ has a solution. Otherwise the rotation direction of the vortex will be independent of the height variable z .

Case 2. Baroclinic elliptic or hyperbolic non-EB mode with skew center.

$$\psi = c(x - x_0(z))^2 - \frac{f}{2}(y - y_0)^2 + \psi_0(z), \quad (30)$$

$$u = f(y - y_0), \quad v = 2c(x - x_0(z)), \quad (31)$$

$$p = p_0(z) + \frac{1}{2}f(2c + f)(y - y_0)^2, \quad (32)$$

where $x_0(z)$, $\psi_0(z)$ and $p_0(z)$ are arbitrary functions of z and y_0 and c are arbitrary constants. If we make the exchanges of $\{x, x_0, u\} \leftrightarrow \{y, y_0, -v\}$ in (30)-(32), the solution is still correct. When $c < 0$, the solution (30)-(32) is related to the baroclinic elliptic non-EB circulation while the baroclinic hyperbolic non-EB mode governed by $c > 0$.

Fig. 5 shows us a special type of structures of (30)-(32) for the vector velocity field with the parameter and function selections

$$f = \frac{1}{2}, \quad c = -1, \quad y_0 = 0, \quad x_0(z) = z^2. \quad (33)$$

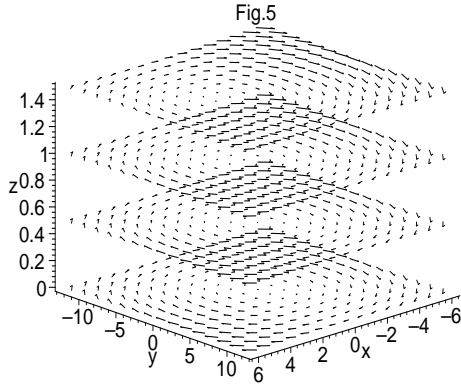


FIG. 5: The structure of the baroclinic elliptic circulation with skew center for the velocity field (31) and the parameter selection (33).

Different from the first type of baroclinic non-EB modes shown by (25)–(27), $\{x_0, y_0\}$, the center of the second type of the baroclinic non-EB circulation (30)–(32), is changeable as the height changes while the length of axes of the circulation is independent of z . That means the circulation has fixed shape with skew center. On the other hand, the pressure and the density distributions have no circulation structure though the stream function and the velocity field do. The rotation direction of the vortex is always independent of the height variable.

4. Summary and discussions. In summary, the fluid systems on the earth such as the oceans and atmosphere have to be studied under the rotational coordinates, the laminar modes for the rotating stratified flows have not yet been studied well though the related topics are successfully studied for the usual irrotational fluid systems.

In this letter, we have studied some types of steady baroclinic equivalent and nonequivalent barotropic modes for rotating stratified flows for the Lilly-Sun model [7, 8]. Usually, to find some exact baroclinic solutions for the rotating fluids is very difficult and only some quite special solutions are found. Using the proper definitions, all the possible baroclinic EB models are obtained. The first type of baroclinic EB modes are determined up to

an arbitrary symmetric Poisson equation which allowed us to get infinitely many exact solutions including vortex street like solutions. The second type of baroclinic EB solutions are quite free because of the existence of an arbitrary stream function with two arbitrary variables. This kind of solutions exhibit rich structures of the jet modes and the symmetric circulations including some hurricane like structures.

In addition to the abundant symmetric circulations all the possible (two types of) elliptic circulations and/or hyperbolic modes are found. It is interesting that the discovery of this kind of solutions disproves Sun's conjecture [7] because they are baroclinic and non-EB modes. For the first type of nonsymmetric circulations, the length of the elliptic axes, the rotation directions, may be changed with respect to the height z while their circulation center is independent of z . For the second type of nonsymmetric circulations, the length of the elliptic axes and the rotation directions are independent of z while the circulation center may skewed as the change of z .

The studies on the vortex solutions of the fluid systems are useful not only in fluid (including atmospheric and oceanic) dynamics but also in many other physical fields including the condense matter, plasma physics, nuclear physics, astrophysics and cosmology [10, 11, 12].

In this paper, rich types of steady modes for rotating stratified flows have been obtained because of the entrance of some arbitrary functions. Though there are also abundant circulations in the nature such as the hurricanes, tornados, ocean circulations etc., we hope that experimental scientists will find some of exact modes mentioned in this letter.

Acknowledgement

The authors are indebt to thanks Dr. X. Y. Tang, Prof. C. Sun and Prof. F. Huang for their helpful discussions. The work was supported by NNSFC (Nos. 10475055, 10601033 and 40305009) and NBRPC (973 Program 2007CB814800).

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